Integrable version of Burgers equation in magnetohydrodynamics

P. Olesen*

Niels Bohr Institute, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark

(Received 7 April 2003; published 23 July 2003)

It is pointed out that for the case of (compressible) magnetohydrodynamics (MHD) with the fields $v_y(y,t)$ and $B_x(y,t)$, one can have equations of the Burgers type which are integrable. We discuss the solutions. It turns out that the propagation of the nonlinear effects is governed by the initial velocity (as in Burgers case) as well as by the initial Alfvén velocity. Many results previously obtained for the Burgers equation can be transferred to the MHD case. We also discuss equipartition $v_y = \pm B_x$. It is shown that an initial localized small scale magnetic field will end up in fields moving to the left and the right, thus transporting energy from smaller to larger distances.

DOI: 10.1103/PhysRevE.68.016307 PACS number(s): 47.65.+a, 05.45.−a

The Burgers equation [1] has been much studied. It can be applied to a variety of phenomena, see, e.g., Refs. [2–4]. Although this equation satisfies a number of properties which are similar to hydrodynamics, it is known to be integrable. Hence, the Burgers equation does not have the properties characterizing chaotic dynamical systems. However, to some extent such properties may be simulated by random boundary conditions [5,6]. Also, the long time behavior of decaying solutions of the Burgers equation with an initial velocity which is homogeneous and Gaussian has been studied and many interesting properties of the spectrum have been found [7]. Also, a general statistical theory of the Stochastic Burgers equation in the inviscid limit has been developed [8]. For a recent review of "burgulence," we refer to the paper by Frisch and Bec [9]. These results make it clear that a number of properties of the Burgers equation are highly nontrivial. With this in mind, we present an integrable generalization of the Burgers equation. The new equations are related to magnetohydrodynamics (MHD) in a way which is analogous to the relation between the Burgers equation and the Navier-Stokes equations. Essentially, all properties found for the Burgers equation can be applied to the new equations, but they also contain some new features.

MHD in $(1+1)$ dimension has been considered by Thomas [10] long time ago. The fields have the velocity $v_x(x,t)$ and the magnetic field $B_x(x,t)$, which satisfy

$$\partial_t v_x + v_x \partial_x v_x = B_x \partial_x B_x + \nu \partial_x^2 v_x,$$

$$\partial_t B_x + v_x \partial_x B_x = B_x \partial_x v_x + (1/\sigma) \partial_x^2 B_x. \tag{1}$$

The first of these equations is similar to the Burgers equation. Both the equations are modeled from the incompressible MHD equations. However, since incompressibility leads to triviality in $(1+1)$ dimensions, equation div $\mathbf{v}=0$ is not enforced, and the total (including the magnetic) pressure is considered to vanish. Also, in these equations it is implicit that variation of the density is disregarded. Similarly, equation div $\mathbf{B}=\mathbf{0}$ is not satisfied. It has been shown by Passot [11] that the above equations are not integrable. For a review of some of the consequences of these equations, we refer to Ref. [12].

In the following, we consider a different form of $(1+1)$-dimensional MHD, where there are really two dimensions, x and y, but where the fields are restricted to depend only on y. We restrict ourselves to fields $B_x(y,t)$ and $v_y(y,t)$. The equation of motion[1]

$$\partial_t \mathbf{v} + (\mathbf{v}\boldsymbol{\nabla})\mathbf{v} = (\boldsymbol{\nabla}\times\mathbf{B})\times\mathbf{B} + \nu\, \nabla^2 \mathbf{v}, \tag{2}$$

then becomes

$$\partial_t v_y + v_y \partial_y v_y = -B_x \partial_y B_x + \nu \partial_y^2 v_y. \tag{3}$$

Similarly, the equation

$$\partial_t \mathbf{B} = \boldsymbol{\nabla}\times(\mathbf{v}\times\mathbf{B}) + (1/\sigma)\nabla^2 \mathbf{B}, \tag{4}$$

where σ is the conductivity of the fluid, becomes

$$\partial_t B_x = -\partial_y (v_y B_x) + (1/\sigma)\partial_y^2 B_x. \tag{5}$$

The coupled set of equations (3) and (5) can be interpreted by saying that Eq. (3) is a Burgers equation with a magnetically generated pressure, governed by Eq. (5). Like in the case of Thomas's $(1+1)$ dimensional equations, div $\mathbf{v}=0$ is not satisfied, and we have also disregarded a possible variation of the density. Notice, however, that div $\mathbf{B}=0$ is satisfied in our approach and that the magnetic pressure $B_x^2/2$ is kept.

The main difference between our equations and those discussed by Thomas is that his Eq. (1) includes a term $B_x \partial_x v_x$ which stretches the magnetic field lines and which competes with the term $v_x \partial_x B_x$ which (in higher dimensions) breaks or twists the field lines. On the other hand, in our case the magnetic field is divergence-free, and we included the magnetic pressure.

*Email address: polesen@nbi.dk

[1] In Eqs. (3) and (5), the density ρ and the vacuum permeability μ_0 should occur. Thus, $(\boldsymbol{\nabla}\times\mathbf{B})\times\mathbf{B}$ on the right-hand side should be multiplied by $1/\mu_0\rho$. We assume a constant ρ. Then, the following rescalings "remove" ρ and μ_0: $\sigma\mu_0 \to \sigma$ and $\mathbf{B}\to\mathbf{B}/\sqrt{\mu_0\rho}$.

The reduced model (1) introduced by Thomas is in correspondence to special subclasses of solutions to the full MHD equations. In our case, Eqs. (3) and (5) are, of course, derived from the real MHD equations, but they do not directly correspond to known subclasses of solutions of the MHD equations. However, the reduced model proposed in this paper may have some physical interest, which we shall discuss at the end of this paper after having given the analytic solution of Eqs. (3) and (5). It turns out that the model predicts that the magnetic energy which is initially localized at small scales is moved to large distances by the nonlinear dynamics.

Conservation of energy can be easily checked from Eqs. (3) and (5). With

$$E_{\rm tot}(t)=\int_{-\infty}^{\infty} dy(v_y^2+B_x^2), \tag{6}$$

one has

$$\frac{dE_{\rm tot}(t)}{dt}=2\int_{-\infty}^{\infty} dy\left(-\frac{1}{3}\partial_y v_y^3-\partial_y(v_yB_x^2)+\nu \; v_y\partial_y^2 v_y + \frac{1}{\sigma}B_x\partial_y^2B_x\right). \tag{7}$$

Assuming no "diffusion at infinity," i.e., assuming that v_y and B_x vanish for $y\rightarrow\pm\infty$, then

$$\frac{dE_{\rm tot}(t)}{dt}=-2\int_{-\infty}^{\infty}dy\left(\nu(\partial_y v_y)^2+\frac{1}{\sigma}(\partial_y B_x)^2\right). \tag{8}$$

Thus, energy is conserved in the limit $\nu,1/\sigma\rightarrow 0$.

The idea is now to compare Eqs. (3) and (5) to the well known solution of the Burgers equation found by Hopf [13] and Cole [14], where the diffusive terms in these equations are included. We can show that if

$$\nu=1/\sigma, \tag{9}$$

then the equations are integrable. We do not know if the equations are still integrable if $\nu\neq 1/\sigma$ and/or if variations of density ρ are included according to the conservation equation

$$\partial_t\rho+\partial_y(\rho v_y)=0. \tag{10}$$

Of course, the full set of equations can be studied numerically.

In the following, we shall consider the case where Eq. (9) is satisfied. It should be emphasized that this assumption is not supposed to represent a realistic estimate of the magnetic Prandtl number $P_m=\sigma\nu$, which, e.g., for liquid metals is of the order[2] of 10^{-5}. Our excuse for having a Prandtl number equal to 1 is primarily that this allows a nontrivial solution of Eqs. (3) and (5). Furthermore, one space dimension is anyhow not realistic.

Adding and subtracting Eqs. (3) and (5), we obtain in the special case (9)

$$\partial_t(v_y+B_x)+\tfrac{1}{2}\partial_y(v_y+B_x)^2=\nu\partial_y^2(v_y+B_x) \tag{11}$$

and

$$\partial_t(v_y-B_x)+\tfrac{1}{2}\partial_y(v_y-B_x)^2=\nu\partial_y^2(v_y-B_x). \tag{12}$$

These two equations are of the Burgers type.

We remind the reader that the solution of the Burgers equation

$$\partial_t u+u\partial_y u=\nu\partial_y^2 u \tag{13}$$

is

$$u(y,t)=\frac{1}{t}[y-\overline{a}(y,t)], \tag{14}$$

where

$$\overline{a}(y,t)=\int_{-\infty}^{\infty} a e^{-[(y-a)^2]/4\nu t+(1/2\nu)\psi(a)}da \Bigg/ \int_{-\infty}^{\infty} e^{-[(y-a)^2]/4\nu t+(1/2\nu)\psi(a)}da. \tag{15}$$

Here, $u(y,t=0)=-\partial_y\psi(y)$. For the case where $\nu=0$, we have

$$u(y,t)=u(b(y,t),t=0), \tag{16}$$

where $b(y,t)$ solves the implicit equation

$$b(y,t)=y-tu(b(y,t),0). \tag{17}$$

Here, $b(y,t)$ can be interpreted as a Lagrangian coordinate, with $b(y,0)=y$ for $t=0$. This solution can be obtained by the methods of characteristics or from the saddle point in Eq. (15) for $\nu\rightarrow 0$. In this case, $b(y,t)\rightarrow\overline{a}(y,t)$. In the case where there are more than one solution of Eq. (17), for $b(y,t)$ one should consider the solution which maximizes the expression

$$-\frac{(y-b)^2}{2t}+\psi(b), \tag{18}$$

with respect to b, as is obvious from the saddle point expansion of Eq. (15).

For the MHD case, the solution can be found in terms of the initial values

$$v_y(y,t=0)\equiv-\partial_y\psi_{v0}(y),$$
$$B_x(y,t=0)\equiv-\partial_y\psi_{B0}(y). \tag{19}$$

The Hopf-Cole solution of Burgers equation then gives

[2]It can be mentioned that in numerical simulations of Earth's dynamo with Prandtl number $\sim 10^{-6}$, one actually uses a value of $P_m\sim 0.1$ (not so far from 1), since this is what is numerically tractable. See, e.g., Ref. [15].

$$v_y(y,t)=\frac{1}{t}\left[y-\frac{1}{2}[\overline{a}_{v+B}(y,t)+\overline{a}_{v-B}(y,t)]\right]$$
$$=-\frac{1}{2}[\overline{\psi'}_{v0+B0}(y,t)+\overline{\psi'}_{v0-B0}(y,t)] \quad (20)$$

and

$$B_x(y,t)=-\frac{1}{2t}[\overline{a}_{v+B}(y,t)-\overline{a}_{v-B}(y,t)]$$
$$=-\frac{1}{2}[\overline{\psi'}_{v0+B0}(y,t)-\overline{\psi'}_{v0-B0}(y,t)]. \quad (21)$$

Here,

$$\overline{a}_{v\pm B}(y,t)$$
$$=\int_{-\infty}^{\infty} a e^{-[(y-a)^2]/4\nu t+(1/2\nu)[\psi_{v0}(a)\pm\psi_{B0}(a)]}da$$
$$\times\left[\int_{-\infty}^{\infty} e^{-[(y-a)^2]/4\nu t+(1/2\nu)[\psi_{v0}(a)\pm\psi_{B0}(a)]}da\right]^{-1}, \quad (22)$$

and similarly,

$$\overline{\psi'}_{v0\pm B0}(y,t)$$
$$=\int_{-\infty}^{\infty}[\psi'_{v0}(a)\pm\psi'_{B0}(a)]$$
$$\times e^{-[(y-a)^2]/4\nu t+(1/2\nu)[\psi_{v0}(a)\pm\psi_{B0}(a)]}da$$
$$\times\left[\int_{-\infty}^{\infty} e^{-[(y-a)^2]/4\nu t+(1/2\nu)[\psi_{v0}(a)\pm\psi_{B0}(a)]}da\right]^{-1}. \quad (23)$$

The last forms of Eqs. (20) and (21) follow from

$$\int_{-\infty}^{\infty}\frac{\partial}{\partial a}e^{-[(y-a)^2]/4\nu t+(1/2\nu)[\psi_{v0}(a)\pm\psi_{B0}(a)]}da=0. \quad (24)$$

In case where ν is considered to nearly vanish, the resulting saddle point simplifies solutions (20) and (21), and $\overline{a}$'s are replaced by solutions of the equations

$$\overline{a}_{v\pm B}=y+t[\psi'_{v0}(\overline{a}_{v\pm B})\pm\psi'_{B0}(\overline{a}_{v\pm B})]. \quad (25)$$

These solutions can also be obtained by solving the original equations without diffusion ($\nu=1/\sigma=0$) by the methods of characteristics. These solutions can be written in a form analogous to Eq. (16),

$$B_x(y,t)=\frac{1}{2}[B_x(b_+(y,t),0)+B_x(b_-(y,t),0)+v_y(b_+(y,t),0)$$
$$-v_y(b_-(y,t),0)] \quad (26)$$

and

$$v_y(y,t)=\frac{1}{2}[v_y(b_+(y,t),0)+v_y(b_-(y,t),0)+B_x(b_+(y,t),0)$$
$$-B_x(b_-(y,t),0)], \quad (27)$$

where $b_\pm(y,t)$ solve the equations

$$b_\pm(y,t)=y-tv_y(b_\pm(y,t),0)\mp tB_x(b_\pm(y,t),0). \quad (28)$$

Like in Burgers case, $b_\pm(y,0)=y$. Also, there is a simple Lagrangian interpretation of Eq. (28), since the right-hand side involves the initial velocity subtracted or added to the usual Alfvén velocity $B_x(b_\pm,0)$, where it should be remembered that constant $1/\sqrt{\mu_0\rho}$ was absorbed in B_x. In the saddle point of Eq. (22) for $\nu\to 0$, we again have $\overline{a}_{v\pm B}(y,t)\to b_\pm(y,t)$, with $b_\pm$ given by the dominant saddle point, as discussed below Eq. (17). It is, of course, easy to show directly that Eqs. (26) and (27) satisfy the original equations (3) and (5) with $\nu=1/\sigma=0$.

From solutions (26) and (27), one can read off a few simple properties. If the initial magnetic field $B_x(y,t=0)$ vanishes, no magnetic field is generated at other times. This is already quite obvious from the original equations (3) and (5). To see this result from the solution by characteristics, we notice that if B_x vanishes at $t=0$, it follows from Eq. (28) that $b_+=b_-$, and Eq. (26) then gives $B_x(y,t)=0$. The velocity field will then behave as a solution of the "pure" Burgers equation for v_y.

A less trivial case is when the initial velocity field vanishes,

$$v_y(y,t=0)=0. \quad (29)$$

Then, we obtain both a magnetic and a velocity field as a consequence of the dynamics, namely,

$$B_x(y,t)=\frac{1}{2}[B_x(b_+(y,t),0)+B_x(b_-(y,t),0)],$$

$$v_y(y,t)=\frac{1}{2}[B_x(b_+(y,t),0)-B_x(b_-(y,t),0)], \quad (30)$$

with

$$b_\pm(y,t)=y\mp t\ \ B_x(b_\pm(y,t),0). \quad (31)$$

We see that if the initial magnetic field is constant, no velocity field is generated. However, in general, a varying initial magnetic field is able to generate a velocity field.

In MHD, it has often been discussed whether there is equipartition, $v_y=\pm B_x$, after a long time. In realistic MHD simulations, one does not find equipartition in all cases. A recent study [16] finds that in nonhelical hydromagnetic turbulence in the inertial range the magnetic energy exceeds the kinetic energy by a factor of 2 to 3. The helical case has recently been discussed in Ref. [17]. In our case, it follows, rather trivially, that if the initial fields satisfy equipartition, then this will be true for all times. In general, we see from Eqs. (26) and (27) that equipartition in the exact sense requires

$$B_x(b_-(y,t),0)=v_y(b_-(y,t),0)$$

and

$$B_x(b_+(y,t),0), \quad v_y(b_+(y,t),0) \quad \text{arbitrary}$$

or

$$B_x(b_+(y,t),0)=v_y(b_+(y,t),0)$$

and

$$B_x(b_-(y,t),0), \quad v_y(b_-(y,t),0) \quad \text{arbitrary}, \tag{32}$$

for $v_y=B_x$ and $v_y=-B_x$, respectively. In the first case it follows from Eq. (28) that $b_-=y$. Hence, from the first line in Eq. (32), it follows that $B_x(y,0)=v_y(y,0)$, so the initial fields are equal. Thus, equipartition requires very special initial fields and is not possible for general initial conditions.

The considerations above do not, however, answer the question concerning equipartition after some time has passed. This requires the study of Eq. (31) for $b_\pm$ after the passage of sufficient time. Let us consider the case $v_y(y,0)=0$ and let us take the initial field $B_x(y,0)$ to be localized in a domain D in y. Then, Eq. (31) shows that $b_\pm$ receive nontrivial contributions from $b_\pm\in D$, and these contributions have different signs, i.e., the nontrivial part of y space move to the right and to the left, so the original domain D splits up into right and left moving domains, $y\in D_R$ and $y\in D_L$. After some time has passed these domains have no overlap. Using result (30), we then see that $B_x(y,t)$ also moves to the left (right) with value $B_x(b_-,0)/2$ ($B_x(b_-,0)/2$). Thus, the value of the magnetic field has decreased by a factor of 2. At the same time, it follows from Eq. (30) that the velocity has increased from zero to $\pm B_x(b_\pm)/2$. Therefore, one has equipartition.

If the initial velocity is nonvanishing, it is more difficult to estimate the result from Eqs. (30) and (31). We shall therefore ask what is the tendency after a short time. We start by considering initial fields which are proportional,

$$B_x(y,0)=\lambda v_y(y,0), \tag{33}$$

where λ is some parameter. We now want to solve Eqs. (26)–(28) perturbatively for small t. Assuming that y is not too close to zero, Eq. (28) can be solved approximately,

$$b_\pm(y,t)\approx y-(1\pm\lambda)tv_y(y,0), \tag{34}$$

where the assumption that y does not vanish is needed in order that the second term on the right-hand side is small relative to y. The fields in Eqs. (26) and (27) can then be expanded, using

$$v_y(b_\pm,0)\approx v_y(y,0)[1-t(1\pm\lambda)\partial_y v_y(y,0)] \tag{35}$$

to obtain the results

$$B_x(y,t)\approx\lambda v_y(y,0)[1-2t\partial_y v_y(y,0)],$$
$$v_y(y,t)\approx v_y(y,0)[1-t(1+\lambda^2)\partial_y v_y(y,0)]. \tag{36}$$

We see that the first order changes in the fields do not preserve the initial proportionality. If we start with a magnetic field which is much smaller than the velocity, $\lambda\ll 1$, then the relative change is larger in B_x than in v_y. If, on the other hand, $\lambda\sim 1$, i.e., the initial fields are comparable, then the corrections found in Eq. (36) are of the same order. If λ is large, i.e., the initial magnetic field is much larger than the initial velocity, then the correction to the velocity is larger than the correction to the magnetic field. Thus, it seems that the nonlinearity in the basic equations tends to increase the smallest initial field, for small times.

We mention that the contributions to the energy from the $O(t)$ corrections in Eq. (36) is proportional to

$$t\int_{-\infty}^{\infty} dy v_y(y,0)^2 \ \ \partial_y v_y(y,0)=\frac{t}{3}v_y(y,0)^3\Big|_{-\infty}^{\infty}, \tag{37}$$

which vanishes since $v_y(y,0)$ must approach zero at infinity in order that the energy is finite. Thus, the corrections in Eq. (36) give no contribution to the total energy. In this argument, we have disregarded that the perturbation may not be valid very close to $y=0$.

We have also investigated the situation numerically, taking the initial fields $v_y(y,0)=\sin y$ and $B_x(y,0)=\lambda v_y(y,0)$ with $-\pi/2<y<\pi/2$. We find that after a long time the fields fluctuate considerably. In general, there is no equipartition, except for $\lambda=1$. The (fluctuating) ratio

$$R=\frac{B_x^2}{v_y^2+B_x^2} \tag{38}$$

is maximally of the order of 0.006 for $\lambda=0.2$ after a long time ($t=14$). For $\lambda=0.9$ one gets a maximum R value around 0.37. Finally, for $\lambda=1.1$ one gets a maximal R around 0.5. Of course, if $\lambda=1$ the numerical calculations give equipartition with $R=0.5$ for all y.

It is clear that the usual Burgers shock waves are present in our case too. From Eq. (28), one has that the derivatives of $b_\pm(y,t)$ become infinite for

$$t=t(b_\pm)=-\frac{1}{v_y'(b_\pm,0)\pm B_x'(b_\pm,0)}. \tag{39}$$

In general, there will actually be more shocks than in the "pure" Burgers case, since derivatives of solutions (27) and (26) contain $\partial_y b_+$ as well as $\partial_y b_-$ which are infinite at $t(b_+)$ as well as $t(b_-)$.

In our case the usual conservation form of the Burgers equation are generalized to

$$\partial_t v_y=\partial_y\left(-\frac{1}{2}v_y^2-\frac{1}{2}B_x^2+\nu\partial_y v_y\right). \tag{40}$$

Moreover, we also have

$$\partial_t B_x=\partial_y\left(-v_y B_x+\frac{1}{\sigma}\partial_y B_x\right). \tag{41}$$

In the last equation, one needs, of course, to replace $1/\sigma$ by ν in order to apply the solutions found in this note.

As already mentioned, the many highly nontrivial properties of the Burgers equation are shared by the solutions for v_y and B_x. This follows simply from the fact that Eqs. (11) and (12) are of the Burgers type, so any property previously derived can be applied to fields v_y+B_x and v_y-B_x. Thus, except for accidental cancellations these properties also hold for fields v_y and B_x. To give an example, consider the correlation function

$$C(y,t)=\langle v_y(y,t)v_y(0,t)+B_x(y,t)B_x(0,t)\rangle, \tag{42}$$

where we take homogeneous random fields $v_y(y,t)$ and $B_x(y,t)$ with ensemble averages. $C(y,t)$ can be obtained from the sum of the correlation functions

$$\langle[v_y(y,t)+B_y(y,t)][v_y(0,t)+B_y(0,t)]\rangle \tag{43}$$

and

$$\langle[v_y(y,t)-B_y(y,t)][v_y(0,t)-B_y(0,t)]\rangle. \tag{44}$$

Each of these two correlation functions contain fields that are solutions of the Burgers equations (11) and (12). Hence, we can use known results from burgulence, for example, from Refs. [7,9], to obtain information on $C(y,t)$. The total energy spectrum is then given by

$$E_{\text{tot}}(k,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}C(y,t)e^{iky}dy. \tag{45}$$

The total kinetic and magnetic energy is then given by

$$E_{\text{tot}}(t)\equiv\langle v_y(y,t)^2+B_x(y,t)^2\rangle=\int_{-\infty}^{\infty}E_{\text{tot}}(k,t)dk. \tag{46}$$

By subtracting Eqs. (43) and (44), we can also obtain the correlator

$$\begin{aligned}D(y,t)&=\langle B_x(y,t)v_y(0,t)+B_x(0,t)v_y(y,t)\rangle\\&=\int_{-\infty}^{\infty}F(k,t)e^{-iky}dk.\end{aligned} \tag{47}$$

The function F is related to the Lorentz force

$$F(t)\equiv 2\langle v_y(y,t)B_x(y,t)\rangle=\int_{-\infty}^{\infty}F(k,t)dk. \tag{48}$$

It is now possible to repeat, for example, the arguments in Ref. [7] to obtain information on $C(y,t)$ and $F(y,t)$, if we assume initial fields which are homogeneous and Gaussian with initial spectra of the form

$$E_{\text{tot}}(k,t)=\alpha^2|k|^n e_0(k) \quad\text{and}\quad F(k,t)=\beta^2|k|^n f_0(k). \tag{49}$$

Here, n is the spectral index and $e_0(k)$ is an even and nonnegative function with $e_0(0)=1$ assumed to be even and decreasing faster than any power of k at infinity. The function $f_0(k)$ has similar properties. The energy spectrum at times different from zero can now be analyzed completely as in Ref. [7]. For example, for $1<n<2$ the spectrum $E_{\text{tot}}(k,t)$ has three scaling regions. The first is for very small k's, where the large eddies are conserved and the behavior agrees with the original $|k|^n$ with a time-independent constant. The second region is a k^2 region and the third region is characterized by a behavior k^{-2}, associated with the shocks. The switching from the first to the second region occurs for a k value around $t^{-1/[2(2-n)]}$, whereas the shift to the last region occurs around $1/\sqrt{t}$, except for logarithmic corrections. For $-1<n<1$ there is no inner region and the spectrum develops in a self-similar fashion. For a much more complete description, we refer to the original paper [7]. It would be of interest to see if somewhat similar results are valid in higher-dimensional MHD. It should also be emphasized that Eqs. (11) and (12) are the independent equations, and hence, one has the possibilty to study more general situations than those discussed in the previous literature. For example, fields v_y+B_x and v_y-B_x may be started out with different random initial fields and their spectra will then develop in different ways. We have already seen an example of this phenomenon in the perturbative calculation.

Concerning the use of results obtained from the study of the Burgers equation we mention that in Eq. (3), one can add a forcing term f on the right-hand side. A very interesting study of the forced Burgers equation in Ref. [8] can then be used in Eqs. (11) and (12), which would have f on the right-hand side. The master equation for the probability density functions of v_y+B_x and v_y-B_x, their differences and gradients, can then be derived as in [8]. Again we refer to the literature [8] for more information.

We now return to the question as to whether our proposed equations have some physical relevance, disregarding the obvious restrictions due to the low-dimensional structure. Our approach has the property that although the fields depend only on one dimension y, the magnetic field $B_x(y)$ points in a different direction. Thus, we can construct an initial state where the magnetic field in the x direction is localized in y and ask how this field propagates from the equations of motion. We take the initial field to be

$$\begin{aligned}&B_x(y,0)=B_0=\text{const}\quad\text{for}\quad -L<y<L,\\&B_x(y,0)=0\quad\text{otherwise},\\&v_y(y,0)=0\quad\text{for all}\quad y,\end{aligned} \tag{50}$$

which is a small scale localized field if L is not too large. This field is a rudimentary version of a flux tube, which realistically would be a magnetic field locally pointing, e.g., in the x direction with a cross section in the y,z plane. In our case, this cross section degenerates to a line segment $-L<y<L$. Ignoring diffusion, the time-dependent solution can be obtained from Eqs. (30) and (31),

$$B_x(y,t)=\frac{1}{2}B_0,\quad v_y(y,t)=\frac{1}{2}B_0 \tag{51}$$

$$\text{for} \quad -L+tB_0<y<L+tB_0,$$

$$B_x(y,t)=\frac{1}{2}B_0, \quad v_y(y,t)=-\frac{1}{2}B_0$$

$$\text{for} \quad -L-tB_0<y<L-tB_0,$$

$$B_x(y,t)=v_y(y,t)=0 \quad \text{outside these intervals.}$$

Thus, the initial flux ‘‘tube’’ decays into two new tubes, which move towards the left and the right. Also, half of the initial magnetic energy is converted into kinetic energy, and equipartition is actually obtained. We see that the original localized configuration turns into a less localized configuration. This example can be generalized to more complicated initial localized fields where B_x has different values in different nearby y intervals. In such cases again the B_x fields in each interval start to split and move out to larger distances to the left and the right. In this way, a fairly localized initial magnetic field will end up as a rather delocalized state, moving energy from small to large distances. This is similar to the phenomenon of an inverse cascade in higher dimensions. This effect has been observed in two and three dimensions in realistic simulations of MHD. Since this phenomenon occurs in our reduced model one can say that our model given by Eqs. (3) and (5) has some physical relevance. However, in three (but not two) dimensions, the inverse cascade is usually linked to helicity, which does not exist in our case for obvious reasons.

There are, of course, very important differences between MHD in one and in higher dimensions. For example, in the latter case (differential), rotation is possible and one can have the dynamo effect. Further, important topological changes of flux tubes are possible. Also, the higher-dimensional MHD equations exhibit genuine chaotic behavior [12], which can only be simulated to some extent in (1+1) dimension by having random initial fields.

I thank Axel Brandenburg for informing me about the work in Ref. [12].

[1] J.M. Burgers, Kon. Ned. Akad. Wet. Verh. **17**, 1 (1939).
[2] G.B. Whitham, *Linear and Nonlinear Waves* (Wiley, New York, 1974).
[3] S.N. Gurbatov, A.N. Malakhov, and A.I. Saichev, *Nonlinear Random Waves and Turbulence in Nondispersive Media* (Manchester University Press, Manchester, 1991).
[4] S.F. Shandarin and Y.B. Zeldovich, Rev. Mod. Phys. **61**, 185 (1989).
[5] Y.G. Sinai, Commun. Math. Phys. **148**, 601 (1992).
[6] Z.S. She, E. Aurell, and U. Frisch, Commun. Math. Phys. **148**, 623 (1992).
[7] S.N. Gurbatov, S.I. Simdyankin, E. Aurell, U. Frisch, and G. Toth, J. Fluid Mech. **344**, 339 (1997).
[8] E. Weinan and E. vanden Eijnden, Commun. Pure Appl. Math. **53**, 852 (2000).
[9] U. Frisch and J. Bec, in *Proceedings of the Les Houches Summer School of Theoretical Physics 2000*, edited by M. Lesieur (Springer-Verlag, Berlin, in press), e-print nlin.CD/0012033.
[10] J.H. Thomas, Phys. Fluids **11**, 1245 (1968).
[11] T. Passot, Phys. Lett. A **118**, 121 (1986).
[12] A. Pouquet, in *Proceedings of the 1987 Les Houches Lectures*, edited by J.P. Zahn and J. Zinn-Justin (North-Holland, Amsterdam, 1993), p. 141.
[13] E. Hopf, Commun. Pure Appl. Math. **3**, 201 (1950).
[14] J.D. Cole, Q. Appl. Math. **9**, 225 (1951).
[15] A. Gerberth, O. Lielausis, E. Platacis, G. Gerbert, and F. Stefani, Rev. Mod. Phys. **74**, 973 (2002).
[16] N.E.L. Haugen, A. Brandenburg, and W. Dobler, e-print astro-ph/0303372.
[17] A. Brandenburg, Astrophys. J. **550**, 824 (2001).